The integral in (8) may be, again, expressed in terms of elliptical integrals; however the related formulas are not adduced here owing to their unwieldiness.

As a very simple particular case of the last problem we shall consider the expansion of a circular cylinder which at the initial instant rotates as a solid at angular velocity ω. Calculations by formula (8) yield

$$F_{11} = F_{22} = (1 + Et^2)^{1/2} \cos [E^{-1/2} \omega \operatorname{arctg} (E^{1/2}t)]$$
$$F_{21} = F_{12} = (1 + Et^2)^{1/2} \sin [E^{-1/2}\omega \operatorname{arctg} (E^{1/2}t)]$$

It is not difficult to compute the variation of the (distribution) density of the cloud moment of momentum during expansion. For a Gaussian initial distribution of the gas density $\rho(r, 0) = (2\pi)^{-1} \exp(-{}^1/_2 r^2)e$ we have

$$(xv_y - yv_x)\rho(r, t)\, 2\pi r\, dr = \omega \exp\left(-\frac{1}{2}\frac{r^2}{1+Et^2}\right)\frac{r^3\, dr}{(1+Et^2)^2}$$

The total momentum $J_{xy} = 2\omega$ obviously remains unchanged. The radius of the layer carrying the maximum moment of momentum varies in time according to the law

$$r_m = [3(1 + Et^2)]^{1/2}$$

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CERTAIN SIMILARITY RELATIONSHIPS FOR THE MOTION OF A GRANULAR COMPACTING MEDIUM

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The solution of one-dimensional self-similar problems of shock wave convergence and of expansion of gas-filled cavities is presented. Conditions of unlimited buildup is derived, two kinds of cavity expansion modes are shown to exist, and the similarity relationship of auger-hole blasting in a uniformly-compacting granular medium and in a fissured rock formation is established.

A class of one-dimensional self-similar problems exists for strong shock waves in gas: concentrated explosion, buildup, short-duration impact, and other (solutions and extensive references are given in [1, 2]). Dimensional considerations indicate that there must exist a similar class of motions in the case of a porous substance consisting of incompressible particles with "dry" friction, but capable of compacting k times under any pressure. For example, the medium referred to in [3] reduces under high loads to such pattern. Dry friction is, in fact, a property of soft soils [4] and shattered rock formations. Certain of the results partly follow from more general (not self-similar) solutions [5, 6, 3].

1. Input equations. Behind a shock wave front bounding the region of motion in problems of explosion and buildup the substance is incompressible, and for one-dimensional flows the following equations are valid:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu(p - p_\varphi)}{\rho r} = 0 \tag{1.1}$$

$$vr^\nu = v_1 R^\nu \tag{1.2}$$

At the front

$$\rho = k\rho_0, \qquad p_1 = \frac{\rho}{k-1} v_1^2, \qquad \frac{dR}{dt} = \frac{k}{k-1} v_1 \tag{1.3}$$

Here t is the time, r the distance from the center (axis, plane) of symmetry, R is the coordinate of the front, ρ and ρ_0 are the final and initial densities, respectively, v is the mass flow rate, $p \equiv p_r$ and p_φ are pressures in the radial and angular (tangential) directions, respectively; $\nu = 2, 1, 0$ for a sphere, a cylinder and a plane, respectively. Subscript 1 relates to the front.

According to [5, 6] in a granular medium (in a limit stress condition) $p_r = ap_\varphi$ when the motion is toward the center (axis of symmetry) and $p_\varphi = ap_r$ for a motion in the reverse direction (for a sphere $p_\theta = p_\varphi$); and $a = (1 - \sin\chi)(1 + \sin\chi)^{-1}$, where χ is the angle of internal friction.

We introduce dimensionless variables

$$y = r / R, \qquad P(y) = p / p_1, \qquad V(y) = v / v_1 \tag{1.4}$$

We shall call the number n in relationships

$$p_1 \sim R^{-(n+\nu+1)}, \qquad v_1 \sim R^{-(n+\nu+1)/2}, \quad E \sim R^{-n} \tag{1.5}$$

the self-similarity index. In these relationships E is the characteristic energy of motion (the kinetic energy of the region of dimension R).

2. Shock wave buildup. After the change to dimensionless variables (1.4) and the elimination of v by means of (1.2) and (1.3), Eq. (1.1) for $p_\varphi = a^{-1}p$ (motion toward the center) becomes

$$\frac{dP}{dy} + \nu\left(1 - \frac{1}{a}\right)\frac{P}{y} = \frac{k(n+1-\nu)}{2y^\nu} + \frac{\nu(k-1)}{y^{2\nu+1}} \tag{2.1}$$

For $P(1) = 1$ its integral is

$$P = \left[\frac{k + a^{-1}}{1 + a^{-1}} + \frac{k(n+1-\nu)}{2(\nu a^{-1} - 1)}\right] y^{\nu(a^{-1}-1)} + \frac{k(n+1-\nu)}{2(1-\nu a^{-1})} y^{1-\nu} - \frac{k-1}{1+a^{-1}} y^{-2\nu} \tag{2.2}$$

The self-similarity index is found from the condition of boundedness of p at large y. Equating to zero the coefficient of the first term, we obtain

$$n = -\frac{2(\nu - a) + ka(1-a)(\nu+1)}{ka(1+a)} \tag{2.3}$$

The complete solution is written as

$$p = \frac{A}{1+a} R^{-(n+\nu+1)}\left[(1+ka)\left(\frac{R}{r}\right)^{\nu-1} - a(k-1)\left(\frac{R}{r}\right)^{2\nu}\right] \tag{2.4}$$

$$|v| = \left((k-1)\frac{A}{\rho}\right)^{1/2}\left(\frac{R}{r}\right)^{\nu} R^{-(n+\nu+1)/2} \tag{2.5}$$

The scale of the phenomenon is determined by A ($A = p$ for $r = R = 1$). Time to the instant of focusing is related to R by the equality

$$|t - t_0| = \frac{2}{k(n+\nu+3)}\left(\frac{(k-1)\rho}{A}\right)^{1/2} R^{(n+\nu+3)/2} \tag{2.6}$$

For all $1 < k < \infty$ and $0 < a < 1$ the motion is accompanied by loss of energy. Dissipation takes place throughout the volume owing to friction (in a sphere and a cylinder) and at the wave front.

An unbounded increase of pressure and velocity takes place prior to focusing, if

$$k > k_0, \qquad k_0 = \frac{\nu - a}{a^2(\nu+1)} \tag{2.7}$$

For $k < k_0$ the solution defines a slowing down motion coming to a full stop at $t = t_0$. The condition necessary for the existence of self-similar modes is in this case $p = 0$ for $r \gg R$ (and not simply boundedness).

In the plane case (2.7) is not satisfied, but trivial solution (2.2) exists with undamped motion: $n = -1$, $P(y) \equiv 1$ and constant rate of collapse.

3. Displacement of medium by gas. Let the medium be compressed by a piston which at instant $t - t_0 = 0$ is at $r = 0$, and whose subsequent motion is such that the pressure on it is a power function of coordinate r_2. An adiabatically expanding gas may, for example, act as such piston. For the adiabatic exponent γ the pressure on the piston is

$$p_2 \sim r_2^{-\gamma(\nu+1)} \tag{3.1}$$

Comparison of (3.1) and (1.5) and the requirement that $p_2 / p_1 =$ const yield

$$n = (\nu+1)(\gamma-1) \tag{3.2}$$

Let us calculate $P_2 = p_2 / p_1$. At the piston

$$y_2 = \frac{r}{R} = \left(\frac{k-1}{k}\right)^{1/(\nu+1)}$$

Substituting $y = y_2$ into (2.2) and a for a^{-1} (a motion away from the center), after simple transformations we obtain

$$P_2(1+a)\left(\frac{k-1}{k}\right)^{\frac{\nu(1-a)}{\nu+1}} = a + k\left[1 - \left(\frac{k-1}{k}\right)^{\frac{1-\nu a}{\nu+1}}\right]\left[1 - \frac{(n+1-\nu)(1+a)}{2(1-\nu a)}\right] \tag{3.3}$$

It is seen that for given k, a and ν the volume of P_2 diminishes with increasing n (i.e. with increasing γ). Letting $P_2 = 0$, we find the limit value of the index

$$n_0 = \frac{(\nu+1)(1-a)}{1+a} + \frac{2a(\nu a - 1)}{(1+a)k}\left[\left(\frac{k}{k-1}\right)^{\frac{\nu a - 1}{\nu+1}} - 1\right]^{-1} \tag{3.4}$$

The index n_0 determines the attenuation of energy for $\gamma \geqslant \gamma_0$, where

$$\gamma_0 = 1 + \frac{n_0}{\nu+1} \tag{3.5}$$

Thus n depends on ν and γ only when $\gamma < \gamma_0$, while for $\gamma \geqslant \gamma_0$ it depends on ν, a and k (it is independent of γ).

We note that the limit self-similarity index also appears in problems concerning shock waves in gas. There the limit modes correspond at rapid piston deceleration to the case of constant energy, or, if the gas fills a half-space, to that of the short-duration impact mode.

For any $1 < k < \infty$ and $0 < a < 1$ the values of n_0 lie within the limits $0 < n_0 < \nu + 1$. Maximum is attained at any a when $k \to \infty$, and for any k when $a = 0$. In a plane case friction is absent and $n_0 = 1$ for any k.

The case of $n_0 = \nu + 1$ corresponds to energy attenuation with conservation of momentum (in small angle sections). For $k \to \infty$ any interaction between sections is absent owing to the zero-thickness of the built-up substance, while at $a = 0$ this is due to $p_\varphi = 0$.

The asymptotic expression of n_0 for a compacting fluid ($a = 1$) are for a sphere and a cylinder, respectively

$$n_0 = \frac{(k-1)^{1/3}}{k^{4/3} - k(k-1)^{1/3}}, \qquad n_0 = \frac{2}{k \ln[k/(k-1)]}$$

The corresponding expressions for $n_0\,(a)$ at the limit $k \to 1$ are of the form

$$n_0 = \begin{cases} 3 - 4a & (0 < a < 1/2) \\ 3(1-a)/(1+a) & (1/2 < a < 1) \end{cases} \qquad n_0 = 2(1-a)$$

Values of n_0 for several k and a in the spherical case are

$k_0 = 1.2$	1.1	1.05	1.02	1.0	—
$n_0 = 2.55$	2.48	2.42	2.36	2.0	$(a = 1/5)$
$n_0 = 2.27$	2.15	2.05	1.96	1.5	$(a = 1/3)$
$n_0 = 1.93$	1.70	1.55	1.43	1.0	$(a = 1/2)$
$n_0 = 1.02$	0.74	0.54	0.36	0	$(a = 1)$

we note that $n_0 \to 3$ when $k \to \infty$, and $n_0 = 3$ when $a = 0$.

Equations (2.5), (2.6), and $p = p_1\,P(y)$ constitute the complete solution of the problem. $P(y)$ is taken from (2.2) in which a is substituted for a^{-1}.

The class of self-similar solutions can be extended, if one considers the piston motion of buildup in an inhomogeneous medium with $\rho_0 \sim r^{\varkappa}$, as was done in [7] with respect to gas.

4. Similarity transformation for auger-hole blasting. The self-similarity index in piston problems is positive. For $R \to 0$ the energy increases, since

$$E = BR^{-n} \tag{4.1}$$

Here B is a parameter which determines the scale of the phenomenon. The applicability of (4.1) is actually limited by the size of the initial bubble and the medium compressibility at high pressures.

Let us consider the motion of a medium under the action of an explosive charge. In the proximity of the products of explosion the medium can be compressible, but at lower loads it conforms to the model considered here. Let the products of explosion at high expansion be a gas with constant γ. We shall establish the relation between the explosion energy E_0 and parameter B in (4.1). By virtue of the gasdynamical similarity law (neglecting thermal conductivity) we have at all stages of motion

$$E = E_0 \left(\frac{E_0^{1/(\nu+1)}}{GR} \right) \tag{4.2}$$

Here G is a certain dimensional parameter depending on the medium properties and on the kind of explosive. For an axisymmetric explosion E and E_0 are energies per unit of length.

In the stage of self-similar motion (4.1) is valid and (C is a number)

$$F\left(\frac{E_0^{1/(\nu+1)}}{GR}\right)=\frac{CE_0^{n/(\nu+1)}}{G^nR^n} \tag{4.3}$$

Hence

$$B=\frac{CE_0^{1+n/(\nu+1)}}{G^n} \tag{4.4}$$

The motion in an unbounded medium has been considered so far. Let the medium border on a void with the interface defined by function $r/h=f_1(\theta,\varphi)$, where h is the distance from the center of explosion in a given direction (e.g. vertical). The motion takes place in a gravitational field with acceleration g. Emergence of the wave over the surface will result in the ejection of substance, followed by the fall of the latter and the formation of a new profile. We shall consider the fallen substance as being compacted.

In the self-similar mode the motion depends on two dimensionally independent parameters [1]. Hence the auger-hole blasting problem contains four defining dimensional parameters: g, h, ρ_0 and, e.g. $\varepsilon=Bh^{-n}$ (a magnitude of the dimension of E_0). We combine these into a dimensionless parameter

$$\mu=\frac{\varepsilon}{\rho_0 g h^{\nu+2}} \tag{4.5}$$

Taking into consideration (4.4), we obtain

$$\mu=\frac{CG^n}{\rho_0 g}\left(\frac{E_0}{h^\alpha}\right)^\beta, \qquad \beta=1+\frac{n}{\nu+1}, \qquad \alpha=\frac{(\nu+1)(n+\nu+2)}{n+\nu+1} \tag{4.6}$$

The final profile of the surface, when presented in the form $r/h=f_2(\theta,\varphi)$, must depend on the dimensionless parameters μ, k and a (or χ) of the problem and on the dimensionless function f_1.

This shows that for similar initial profiles the profiles of craters produced by an explosion will be similar, if the explosion energy is proportional to the αth power of the depth of the charge deposition. This result also holds when the density, porosity, and the coefficient of internal friction vary away from the center of explosion, provided that the similarity of their distribution remains unaltered at explosions of different magnitude.

The dependence of α on k and a is weak. Since $0<n<\nu+1$, hence

$$\nu+{}^3/_2<\alpha<\nu+2$$

At strong dissipation of energy α is close to its lower limit.

In a moist soft medium $a\approx 0.5$ [4], while in a dry soft one and in a crushed rock formation friction is higher. If ${}^1/_5<a<{}^1/_2$ and $1.02<k<1.2$ (to which corresponds $1.43<$ $<n_0<2.55$), then for a spherical explosion with $\gamma\gg\gamma_0$ we have $3.54<\alpha<3.67$. For $n_0=2$ we have $\alpha=3.6$ and $\gamma_0={}^5/_3$. If $\gamma<\gamma_0$, then $\alpha=(3\gamma+1)/\gamma$.

Since dry friction is defined by dimensionless coefficients, the derivation of the similarity relationships was possible without detailed consideration of the unsymmetric motion in auger-hole blasting. The presence at this stage of complex dissipative processes is taken into consideration in the results presented here.

On the assumption of identical compacting ability of the medium at any pressure, the deformation of the profile will, strictly speaking, continue after the fall and crumbling of the soil. Crater profiles are similar at corresponding instants of time (e.g. at the instant of "landing" of the last grain of sand). When $k \to 1$, any subsequent motion is absent.

It is important to note that all conclusions about the existence of self-similar modes in the ejection of a medium by gas and, as a consequence, the power law of similarity expressed by $E_0 \sim h^\alpha$ remain valid for auger-hole blasting, even when ρ / ρ_0 and the effective value of χ behind the wave front are not constants, but functions of relative strain. This follows from dimensional considerations. A similar model is valid, for example, for approximately defining an explosion in a strongly fissured rock taking into account its gradual transformation into detritus. We would add that the existence of self-similar modes with expansion of a small cavity does not necessarily require the presence of a compression jump. An example of this is the expansion of a bubble in an incompressible fluid (the second stage in the Rayleigh problem).

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ON THE PROBLEM OF GLIDING OF A PLATE ON THE SURFACE OF A HEAVY IDEAL LIQUID OF FINITE DEPTH

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The two-dimensional problem of the flow around an arbitrary contour floating on the surface of a heavy ideal fluid of finite depth is considered. By using the results of [1 - 3] the problem mentioned is reduced by operational calculus methods in Sect. 1 to the determination of the pressure on the contour from an integral equation of the first kind with nonregular difference kernel of complex structure dependent on two dimensionless parameters λ and δ.

The case of gliding of an inclined plate is studied in detail in Sects. 2-4. An asymp-